EXISTENTIALLY CLOSED STRUCTURES AND JENSEN'S PRINCIPLE ◇

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ABSTRACT
Assume $V = L$, or even $\diamond_{\aleph_1}$, there is no uncountable locally finite group which can be embedded in every uncountable universal locally finite group. Similar results hold for existentially closed groups and division rings.

0. Introduction

In [8] Shelah and I proved that in each cardinal $\kappa > \aleph_0$ there are 2^κ non-isomorphic universal locally finite groups. This answered a question of Kegel and Wehrfritz [7]. In fact we provided rather more detailed algebraic information. Namely, there is a locally finite group H of cardinal $\aleph_1$ such that in each cardinal $\kappa > \aleph_0$ there is a universal locally finite group G which has no subgroup isomorphic to H. This gives two non-isomorphic universal locally finite groups of cardinal κ, since it is known on general grounds that there is a universal locally finite group of power κ with H as a subgroup.

We found such an H which is 2-step solvable and of exponent 6. We wondered if H could be chosen abelian. Philip Hall [5] had shown that all countable locally finite groups are embeddable in all universal locally finite groups, so we posed the following

PROBLEM. Which locally finite groups H are embeddable in all universal locally finite groups of cardinal $\geqq \text{card}(H)$?

Let us call such groups *inevitable*. In the present paper I contribute to the above problem the following:

THEOREM. *Assume $V = L$. There are no inevitable abelian groups of cardinal $\aleph_1$.*

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So, assuming $V = L$, the 2-step solvable group mentioned above can be replaced by *any* locally finite abelian group of cardinal $\aleph_1$.

I do not know if $V = L$ is needed for this.

The method applies not only to universal locally finite groups but also to existentially closed groups and division rings [6]. I give a uniform presentation for all three classes.

Note added in proof, August 18, 1976. Much stronger results have now been obtained for groups and locally finite groups. Independently of my work, Kenneth Hickin of Michigan State proved Theorem 1 for locally finite groups using only *ZFC*. Somewhat later, at the group theory meeting in Oxford, in July 1976, Shelah proved Theorem 1 for groups, using only *ZFC* + *CH*. Both workers use much "slower" (and more subtle) enumerations than I do, and in addition Shelah uses some very deep results of Ziegler on e.c. groups. So far there are no improvements in the case of division rings. The analogue of Shelah's result may need some version of Higman's embedding theorem for division rings.

Both workers obtained many other interesting results. For example, Shelah proved that there are no uncountable inevitable locally finite groups.

1. Basic concepts

I refer to [6] for all the model-theoretic ideas needed in this paper.

1.1. Let $\mathscr{C}$ be a class of structures for a countable language $\mathscr{L}$. $\mathscr{C}$ will be either

i) the class of existentially closed groups,

ii) the class of existentially closed division rings,

iii) the class of universal locally finite groups.

Note that (iii) is in fact the class of existentially closed locally finite groups, as pointed out in [8].

I will isolate certain axioms on the class $\mathscr{C}$, and use only these in the sequel.

The first is

(UC): $\mathscr{C}$ is closed under union of chains.

1.2. *Centralizers.* In each of the above cases we have the notion of the centralizer of a set of elements. I will look at this quite abstractly, as follows.

For each $\mathscr{M}$ in $\mathscr{C}$ we have a map

$$X \mapsto C_{\mathscr{M}}(X)$$

from the power set of $\mathscr{M}$ to itself.

$C_{\mathcal{M}}(X)$ is the called the centralizer of X in $\mathcal{M}$. The following axioms are assumed:

(Ord): $$X \subseteq Y \Rightarrow C_{\mathcal{M}}(Y) \subseteq C_{\mathcal{M}}(X)$$

(Lim) Suppose $\langle \mathcal{M}_\lambda : \lambda < \mu \rangle$ is an increasing chain in $\mathscr{C}$, and $\mathcal{M} = \bigcup_{\lambda<\mu} \mathcal{M}_\lambda$. Suppose $X \subseteq \mathcal{M}_0$. Then

$$C_{\mathcal{M}}(X) = \bigcup_{\lambda<\mu} C_{\mathcal{M}_\lambda}(X \cap \mathcal{M}_\lambda).$$

Note that Lim trivially implies that if $\mathcal{M}, \mathcal{N} \in \mathscr{C}$ and $\mathcal{M} \subseteq \mathcal{N}$ and $X \subseteq \mathcal{M}$ then $C_{\mathcal{M}}(X) \subseteq C_{\mathcal{N}}(X)$.

Let me say that X is abelian in $\mathcal{M}$ if $X \subseteq C_{\mathcal{M}}(X)$. By the preceding remark, the phrase "in $\mathcal{M}$" may be dropped.

Ord and Lim are of course true for the classes (i), (ii), (iii) with the conventional interpretation of centralizer.

1.3. Now I impose another axiom on centralizers. This holds for (i), (ii), (iii), but the verification is not trivial. I call the principle BPW after Boffa, Van Praag, and Wheeler, who obtained weaker versions of it for (i) and (ii) in [2, 3, 6].

(BPW) Suppose $\mathcal{M} \in \mathscr{C}$, $\mathcal{M}$ countable. Suppose X_j, $j \in \omega$, are subsets of $\mathcal{M}$ such that no X_j is contained in a finitely generated substructure of $\mathcal{M}$. Then there is a proper countable extension $\mathcal{N}$ of $\mathcal{M}$, $\mathcal{N}$ in $\mathscr{C}$, such that

a) $C_{\mathcal{N}}(X_j) = C_{\mathcal{M}}(X_j)$ for $j \in \omega$,

and

b) X_j is not included in a finitely generated subset of $\mathcal{N}$, for j in ω.

I will prove BPW later later for (i), (ii), (iii), with the standard interpretation of centralizers. In those cases, $\mathcal{N}$ can be chosen isomorphic to $\mathcal{M}$.

2. Applying $\diamondsuit$

I refer to Devlin [4] for all one needs to know on stationary sets and the Jensen combinatorial principles.

2.1. I will be using

($\diamondsuit_{\omega_1}$): There are sets S_α ($\alpha < \omega_1$) such that $S_\alpha \subseteq \alpha$ and for any $X \subseteq \omega_1$

$$\{\alpha < \omega_1 : X \cap \alpha = S_\alpha\} \quad \text{is stationary in } \omega_1.$$

Jensen proved that $\diamondsuit_{\omega_1}$ holds in $ZFC + V = L$.

I write $\diamondsuit$ for $\diamondsuit_{\omega_1}$ in this paper, and fix once and for all sets S_α as above.

2.2. LEMMA 1. *Assume* UC, Ord, Lim, BPW, *and* $\diamond$. *Let* $\mathcal{M} \in \mathcal{C}$, $\mathcal{M}$ *countable. Then there exists* $\mathcal{N}$ *in* $\mathcal{C}$, $\mathcal{M} \subset \mathcal{N}$, $\mathcal{N}$ *of cardinal* $\aleph_1$, *such that every abelian subset of* $\mathcal{N}$ *is countable.*

PROOF. Let λ_α, $\alpha < \omega_1$, be the limit ordinals $< \omega_1$, in increasing order. I may assume that the domain of $\mathcal{M}$ is ω, i.e. λ_0. I will construct an increasing chain $\mathcal{M}_\alpha$, $\alpha < \omega_1$ of members of $\mathcal{C}$ such that

a) $\mathcal{M}_0 = \mathcal{M}$;

b) $\mathcal{M}_\alpha$ has domain λ_α;

c) $\mathcal{M}_\alpha = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$ if α is a limit;

d) for each $\alpha < \omega_1$, and each $\beta \leqq \lambda_\alpha$, either S_β is included in a finitely generated substructure of $\mathcal{M}_\alpha$, or

$$C_{\mathcal{M}_{\alpha+1}}(S_\beta) = C_{\mathcal{M}_\alpha}(S_\beta)$$

and S_β is not included in a finitely generated substructure of $\mathcal{M}_{\alpha+1}$.

The existence of such a chain is immediate from BPW, UC, Ord, and Lim.

Let $\mathcal{N} = \bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha$. Then $\mathcal{N}$ is in $\mathcal{C}$, by UC, and has domain ω_1, whence cardinal $\aleph_1$.

Now let X be any subset of $\mathcal{N}$. By $\diamond$, the set

$$S = \{\beta < \omega_1 : X \cap \beta = S_\beta\}$$

is stationary in ω_1.

Suppose $C_{\mathcal{N}}(X)$ is countable. I will deduce that X is countable.

By Ord, if $\beta \in S$ then $C_{\mathcal{N}}(S_\beta)$ is uncountable. By Lim, there exists $\gamma > \beta$ such that

$$C_{\mathcal{M}_{\gamma+1}}(S_\beta) \neq C_{\mathcal{M}_\gamma}(S_\beta).$$

But then S_β is included in a finitely generated substructure of $\mathcal{M}_\gamma$. But then by our construction S_β is included in a finitely generated substructure of $\mathcal{M}_\beta$.

If β is a limit, S_β must be included in a finitely generated substructure of some $\mathcal{M}_\tau$, $\tau < \beta$.In this case, let $f(\beta) =$ the least such τ. Then $f(\beta) < \beta$.

Let S' consist of the limit ordinals in S. Then S' is stationary. By Fodor's Theorem [4], f is constant on some stationary $T \subseteq S'$. Let $f(\delta) = \gamma_0$, for δ in T. But then $X = \bigcup_{\delta \in T} X \cap S_\delta \subseteq \mathcal{M}_{\gamma_0}$, so X is countable, as required.

This proves the lemma, for if X is abelian and uncountable $C_{\mathcal{N}}(X)$ is uncountable.

3. Establishing BPW

I now prove BPW for classes (i), (ii), (iii). The basic idea is the same in all cases, but (iii) requires a slightly different preliminary treatment, due to the failure of the amalgamation property for locally finite groups [9]. (i) and (ii) can be handled together by minor modifications of the methods of Boffa, Van Praag and Wheeler cited earlier.

3.1. Assume now $\mathscr{C}$ is one of the classesof 1.1. For $a_1, \cdots, a_n \in \mathscr{M} \in \mathscr{C}$, let $\langle a_1, \cdots, a_n \rangle$ be the substructure of $\mathscr{M}$ generated by $a_1, \cdots, a_n$.

LEMMA 2. *Suppose* $\mathscr{M} \in \mathscr{C}$ *and* $a_1, \cdots, a_n, a, b, c \in \mathscr{M}$. *Suppose* $a, b, c \notin \langle a_1, \cdots, a_n \rangle$. *Then there is an inner automorphism* σ *of* $\mathscr{M}$ *such that* $\sigma(a_i) = a_i$, $1 \leqq i \leqq n$,

$$b \notin \langle a_1, \cdots, a_n, \sigma(a) \rangle,$$

and

$$\sigma(a)c \neq c\sigma(a).$$

Moreover, $\sigma(a)$ *can be chosen to lie outside any prescribed finitely generated substructure* Δ *of* $\mathscr{M}$.

PROOF. Let $H = \langle a_1, \cdots, a_n \rangle$, and let G be the substructure generated by $a_1, \cdots, a_n, a, b, c$ and Δ. Let $K = G_{*H}G$, the free product of two copies of G, amalgamating H. The intended meaning of K is clear in cases (i) and (ii). In (iii) g give K the same meaning as in (i), i.e. K is the usual group amalgam. In this case K is not locally finite, so I have to do a special argument below.

In each case, G can be construed as a substructure of K via

$$\begin{array}{c} G_{*H}G \\ \nwarrow \\ G \end{array}$$

Let $\phi: G \rightarrow K$ be the other natural embedding

$$\begin{array}{c} G_{*H}G \\ \nearrow \\ G \end{array}$$

In all cases $\phi(h) = h$ for $h \in H$. From the general theory of free products for groups and fields, one has in all cases

$$b \notin \langle a_1, \cdots, a_n, \phi(a)\rangle,$$

$$\phi(a)\cdot c \neq c\cdot\phi(a),$$

and $\phi(a) \notin \Delta$.

Now I finish the proof for (i) and (ii). Using amalgamation one gets an extension $\mathcal{M}'$ of $\mathcal{M}$, with $\mathcal{M}' \in \mathcal{C}$, and an extension $\psi: K \to \mathcal{M}'$ which is the identity on G. From the general theory of existentially closed groups and division rings [6], it follows that there is an inner automorphism of $\mathcal{M}'$ which restricts to the map

$$g \mapsto \psi\phi(g)$$

on G.

Note that $b \notin \langle a_1, \cdots, a_n, \psi\phi(a)\rangle$,

$$\psi\phi(a)\cdot c \neq c\cdot\psi\phi(a),$$

and

$$\psi\phi(a) \notin \Delta.$$

So there is an element t of $\mathcal{M}'$ such that $b \notin \langle a_1, \cdots, a_n, t^{-1}at\rangle$,

$$t^{-1}at\cdot c \neq c\cdot t^{-1}at,$$

and $t^{-1}at \notin \Delta$.

Select generators $\delta_1, \cdots, \delta_k$ for Δ. By general theory [6], there exists u, v in $\mathcal{M}'$ such that

$$\begin{aligned}
&u^{-1}bu \neq b, && \\
&u^{-1}a_iu = a_i, && 1 \leqq i \leqq n \\
&u^{-1}t^{-1}atu = t^{-1}at && \\
&v^{-1}t^{-1}atv \neq t^{-1}at && \\
&v^{-1}\delta_jv = \delta_j, && 1 \leqq j \leqq k.
\end{aligned}$$

Since $\mathcal{M}$ is existentially closed, $\mathcal{M} <_1 \mathcal{M}'$, so one could have chosen t, u, v in $\mathcal{M}$. Let $\sigma(x) = t^{-1}xt$ and the lemma is proved.

The problem for (iii) is that one cannot embed K in any $\mathcal{M}'$ in $\mathcal{C}$. So one has to replace K by a suitable finite F. This is made possible by results of Baumslag [1] and Neumann [9]. Since G is finite, K is residually finite, by [1, theorem 2]. It follows that there is a finite group F_1 and a homomorphism $bm_1: K \to F_1$ such that μ_1 is a monomorphism on G and on $\phi(G)$, and $\mu_1\phi(a)\cdot\mu_1(c) \neq \mu_1(c)\cdot\mu_1\phi(a)$. Next, take F_2 as a permutational amalgam of two copies of G over H. See [9, p.

479] for details. Let $\mu_2: K \to F_2$ be the canonical epimorphism. By [9, p. 479], F_2 is finite, μ_2 is a monomorphism on G and $\phi(G)$,

$$\mu_2(b) \notin \langle \mu_2(a_1), \cdots, \mu_2(a_n), \mu_2\phi(a) \rangle$$

and $\mu_2\phi(a) \notin \mu_2(\Delta)$.

Let $F = F_1 \times F_2$, and $\mu = \mu_1 \times \mu_2$.

Then F is finite, and μ is a monomorphism on G and $\phi(G)$,

$$\mu(b) \notin \langle \mu(a_1), \cdots, \mu(a_n), \mu\phi(a) \rangle$$

$\mu\phi(a) \cdot \mu(c) \neq \mu(c) \cdot \mu\phi(a)$, and $\mu\phi(a) \notin \mu(\Delta)$.

One can take μ to be the identity on G, so $G \subseteq F$. Now I just reproduce the final argument given for (i) and (ii), but replacing K by F. In fact, I can take $\mathcal{M}' = \mathcal{M}$ since F is finite.

3.2. Now I can prove (BPW), or rather the inside-out version of it.

LEMMA 3. *Suppose $\mathcal{C}$ is* (i), (ii), *or* (iii). *Suppose $\mathcal{M} \in \mathcal{C}$, $\mathcal{M}$ countable, and X_j, $j \in \omega$, are subsets of $\mathcal{M}$ such that no X_j is cntained in a finitely generated substructure of $\mathcal{M}$. Let b be a non-central element of $\mathcal{M}$. Then there is a substructure $\mathcal{M}'$ of $\mathcal{M}$ such that $b \notin \mathcal{M}'$, and an isomorphism $\mathcal{M} \to \mathcal{M}'$ of $\mathcal{M}$ onto $\mathcal{M}'$ such that*

a) $C_{\mathcal{M}}(X_j') = C_{\mathcal{M}'}(X_j')$

where X_j' is the image of X_j, and

b) *X_j' is not contained in any finitely generated substructure of $\mathcal{M}$.*

PROOF. (Following Boffa [2], but with extra complications.)

Put on $\mathcal{M}$ a well-ordering of type ω. When I say "least" I mean with respect to this well-ordering.

Put on the set $\mathcal{F}$ of finitely generated substructures of $\mathcal{M}$ a well-ordering of type ω, as $\Delta_0, \Delta_1, \cdots, \Delta_n, \cdots$. $n < \omega$.

Define inductively a sequence $(a_n)_{n<\omega}$ as follows: a_{2n} is the least element of $\mathcal{M} - \langle a_0, \cdots, a_{2n-1} \rangle$; a_{n+1} is the least element of $X_k - \langle a_s, \cdots, a_{2n} \rangle$, where k is the largest integer j such that 2^j divides $n+1$.

Using Lemma 2, one can define inductively a sequence $\langle a_n' \rangle_{n<\infty}$of elements of $\mathcal{M}$ such that

$$(\alpha) \qquad \langle a_0, \cdots, a_n \rangle \cong \langle a_0', \cdots, a_n' \rangle$$

via an inner automorphism σ_n of $\mathcal{M}$ sending a_i to a_i';

$$(\beta) \qquad b \notin \langle a_0', \cdots, a_n' \rangle;$$

$$(\gamma) \qquad \text{if } n+1 = 2^j \cdot (2l+1),$$

then $a'_{2n+1} \notin \Delta_l$ and a'_{2n+1} does not commute with the first element of $\mathcal{M} - \langle a'_0, \cdots, a'_{2n} \rangle$ which commutes with every member of $\sigma_n(X_j \cap \langle a_o, \cdots, a_n \rangle)$.

Let $\mathcal{M}'$ be the substructure of $\mathcal{M}$ generated by $\langle a'_n\ n \in \omega \rangle$. Clearly $\mathcal{M}' \cong \mathcal{M}$ and $b \notin \mathcal{M}'$.

Fix the isomorphism $\mathcal{M} \to \mathcal{M}'$ given by the union of the σ_n. Let X'_j be the image of X_j. By considering (γ) at all stages $n = 2^j(2l+1) - 1$ for fixed j, one sees that $C_{\mathcal{M}'}(X'_j) = C_{\mathcal{M}}(X'_j)$. Similarly, at $n = 2^j(2l+1) - 1$, one arranges that $X'_j \not\subset \Delta_l$. So X'_j is not included in any finitely generated substructure of $\mathcal{M}$.

This proves the lemma.

Corollary. (BPW) *holds for* (i), (ii), (iii), *and $\mathcal{N}$ can be chosen isomorphic* to $\mathcal{M}$.

Proof. Turn the preceding outside in.

4. The main theorem

Henceforward $\mathcal{C}$ is (i), (ii), or (iii), and centralizer has its usual meaning.

Recall that it is known that if $\mathcal{M} \subset \mathcal{N}$, with $\mathcal{M}, \mathcal{N} \in \mathcal{C}$, and if $\mathcal{M}, \mathcal{N}$ embed the same finitely generated substructures then $\mathcal{M} <_{\infty,\omega} \mathcal{N}$. See [6, 8]. So, by inspecting the proof of Lemma 1 and using the corollary to Lemma 3, one sees that in Lemma 1 one can arrange $\mathcal{M} <_{\infty,\omega} \mathcal{N}$.

Recall also that in Lemma 1 one actually proved that if $C_{\mathcal{N}}(X)$ is uncountable then X is countable.

So one has:

Theorem 1. *Assume $V = L$. Suppose $\mathcal{M} \in \mathcal{C}$, $\mathcal{M}$ countable. Then there exists $\mathcal{N} \in \mathcal{C}$, $\mathcal{M} <_{\infty,\omega} \mathcal{N}$, $\mathcal{N}$ of cardinal $\aleph_1$, such that if $C_{\mathcal{N}}(X)$ is uncountable then X is countable. In particular, every abelian substructure of $\mathcal{N}$ is countable.*

Note. In cases (i) or (ii), if $\mathcal{M}$ is generic for finite (resp. infinite) forcing, then so is $\mathcal{N}$. See [6].

Definition (Fix $\mathcal{C}$). Suppose $\mathcal{A}$ is an $\mathcal{L}$-structure embeddable in a member of $\mathcal{C}$. $\mathcal{A}$ is $\mathcal{C}$-inevitable if $\mathcal{A}$ is embeddable in every member of $\mathcal{C}$ of cardinal $\geqq$ cardinal of $\mathcal{A}$.

Theorem 2. *(Fix $\mathcal{C}$). Assume $V = L$. If $\mathcal{A}$ is abelian of cardinal $\aleph_1$, then $\mathcal{A}$ is not $\mathcal{C}$-inevitable.*

5. Conclusion

I conclude with two obvious problems.

1) Can Theorems 1 and 2 be proved in *ZFC*, or *ZFC* + *CH*?

2) Can $\aleph_1$ in Theorem 1 be replaced by larger cardinals?

It is natural to try to use morasses on the second problem.

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